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Global Description of the Solutions of a Large Class of Non-integrable Hamiltonians

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1. INTRODUCTION

Two degrees of freedom Hamiltonian systems associated to central potentials in the plane, i.e., of the form

$$\begin{aligned} &H_0 : \mathfrak{R}^+ \times S^1 \times \mathfrak{R}^2 \to \mathfrak{R} \\ &(r, \theta, p_r, p_\theta) \to \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - f(r), \end{aligned} \tag{1}$$

are integrable systems that model many simple physical problems.

In fact, (1) is the Hamiltonian of a particle moving in the plane under the action of S^1-symmetric potential field. A less naïve approximation of any physical situation roughly modeled by a system of the form (1) leads to a Hamiltonian of the form

$$\begin{aligned} &H : \mathfrak{R}^+ \times S^1 \times \mathfrak{R}^2 \to \mathfrak{R} \\ &(r, \theta, p_r, p_\theta) \to \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + f(r)\, V(\theta). \end{aligned} \tag{2}$$

System (2) will in general be non-integrable. In the present work, we shall consider Hamiltonians of the form (2) with $f(r) : \mathfrak{R}^+ \to \mathfrak{R}$ a C^2-function subjected to certain assymptotic conditions at $r = 0$, namely, that $f(r) = O(r^{-x})$, $x > 2$ (see Section 2). We have taken the perturbation term $V(\theta)$ of the form $V(\theta) = -(\mu \cos^2 \theta + \sin^2 \theta)^{-x/2}$, $\mu > 1$, which is the generalization

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of the perturbing term of the anisotropic Kepler problem, see [CLl], but the results obtained here may be immediately extended to other negative functions with n non-degenerate maxima and n non-degenerate minima.

Physically, the condition $f(r) = O(r^{-x})$, $x > 2$, means that close to the singularity at $r = 0$ the attracting potential dominates over the centrifugal force. Our main result is to show that, in these conditions, it is possible to classify completely the orbits of (2). More precisely, we construct an invariant partition of the energy levels I_h of (2) such that to each set of the partition correspond orbits with a certain qualitative behaviour (see Section 3, Theorem 3.11).

In particular, our results apply when $f(r)$ is a homegeneous function of degree $k < -2$. In this case, we may use a non-integrability criterium based on Ziglin's Theorem due to Yoshida (see [Z, Y]) and conclude that the Hamiltonian (2) is non-integrable for every $\mu > 1$. Hence, Theorem 3.11 gives a qualitative description of all the orbits of a large family of non-integrable Hamiltonians. Also, since the type of orbits that we find is essentially the same for every function $f(r)$ in the class F that we consider, our result leads to the conjecture that system (2) is non-integrable for every $f \in F$ and every $\mu > 1$.

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2. Equations of Motion and the Regularized Phase Space

Let F denote the set of functions $f: \Re^+ \to \Re$ of class C^2 such that:

$$\lim_{r \to 0} f(r) = +\infty,$$

there exists $x \in \Re^+$ such that $\lim_{r \to 0} f(r) r^x = 1$,

$$\lim_{r \to 0} \frac{d}{dr}\left(\frac{df}{dr}(r) r^{x+1}\right) = 0,$$

$$\lim_{r \to 0} \frac{d}{dr}(r^x f(r)) = 0 \qquad \text{and}$$

$$\lim_{r \to 0} (x f(r) + r f'(r)) = 0.$$

We shall denote by F_α the set of $f \in F$ that verify the above condition with $x = \alpha$.

Consider the Hamiltonian

$$H(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + f(r)\, V(\theta), \tag{2.1}$$

where $V(\theta) = -(\mu \cos^2 \theta + \sin^2 \theta)^{-\alpha/2}$, $\alpha \in (2, +\infty)$, $\mu \in (1, +\infty)$, $(p_r, p_\theta) \in \Re^2$, $(r, \theta) \in \Re^+ \times S^1$, and $f \in F_\alpha$.

The equations of motion of system (2.1) are

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{\theta} &= p_\theta r^{-2}, \\ \dot{p}_r &= p_\theta^2 r^{-3} - \frac{df}{dr}(r)\, V(\theta), \\ \dot{p}_\theta &= -f(r)\, \frac{dV}{d\theta}(\theta). \end{aligned} \tag{2.2}$$

Introducing McGehee's coordinates, [Mc], (r, θ, v, u, τ) given by

$$v = r^{\alpha/2} p_r, \qquad u = r^{(\alpha-2)/2} p_\theta, \qquad \frac{dt}{d\tau} = r^{\alpha/2+1}, \tag{2.3}$$

Eqs. (2.2) become

$$\begin{aligned} r' &= rv, \\ \theta' &= u, \\ v' &= \frac{\alpha}{2} v^2 + u^2 - r^{\alpha+1} \frac{df}{dr}(r)\, V(\theta), \\ u' &= \frac{\alpha-2}{2} uv - r^\alpha f(r) \frac{dV}{d\theta}(\theta), \end{aligned} \tag{2.4}$$

where "$'$" denotes derivative with respect to τ. The energy relation $H(r, \theta, p_r, p_\theta) = h$ written in terms of the new variables is

$$\tfrac{1}{2}(u^2 + v^2) + r^\alpha f(r)\, V(\theta) = r^\alpha h. \tag{2.5}$$

Equations (2.4) provide an analytical extension of the flow of system (2.2) to $r = 0$. In terms of the phase space, (2.4) extends the invariant sets determined by $I_h = \{(r, \theta, v, u) \in \Re^+ \times S^1 \times \Re^2 : H(r, \theta, v, u) = h\}$ to the sets $\{(r, \theta, v, u) \in (\Re^+ \cup \{0\}) \times S^1 \times \Re^2 : H(r, \theta, v, u) = h\}$. From the first equation of (2.4) and (2.5) we obtain that the singularity $r = 0$ on the boundary of an energy level I_h corresponds in the new coordinates (2.3) to the invariant manifold, which we shall call the collision manifold,

$$\Lambda_h = \{(r, \theta, v, u): r=0, \theta \in S^1, (v, u) \in \mathfrak{R}^2 \text{ and } u^2+v^2 = -2V(\theta)\}. \tag{2.6}$$

Note that Λ_h does not depend on the energy h and so it is common boundary for all I_h, $h \in \mathfrak{R}$. Moreover Λ_h is a surface with axial symmetry like the one represented in Fig. 2.1.

Taking the limit $r \to 0$ in (2.4) and using condition $f \in F_\alpha$ and (2.6) we obtain the flow on Λ:

$$\begin{aligned} \theta' &= u, \\ v' &= \left(1 - \frac{\alpha}{2}\right) u^2, \\ u' &= \left(\frac{\alpha}{2} - 1\right) uv - \frac{dV(\theta)}{d\theta}. \end{aligned} \tag{2.7}$$

Proposition 2.1. *Consider system* (2.2), *and suppose that the straightline $y = h$ intersects transversally the graphic of $f(r)$. Then, for μ in a sufficiently small neighbourhood of $\mu = 1$, the energy level I_h has a connected component I_h^* diffeomorphic to an open solid torus whose closure is $\bar{I}_h^* = I_h^* \cup \Lambda$. Furthermore, the flow in $\bar{I}_h^*$ is such that the only equilibrium points are $p^\pm(\theta_0) = (r=0,\ \theta_0 = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ v = \pm(-2V(\theta_0))^{1/2},\ u=0)$ on Λ.*

Proof. If the horizontal line $y = h$ intersects the graphic of $f(r)$ transversally then, for μ sufficiently near to $\mu = 1$ and for every $\theta \in S^1$, $y = h$ will also intersect transversally the graphic of $f(r)\ V(\theta)$. Given the form of the

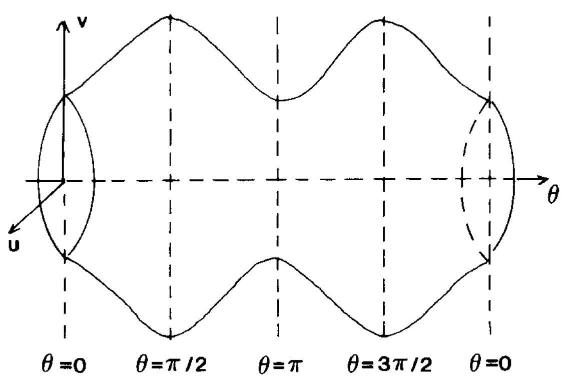

FIG. 2.1. The collision manifold Λ.

Hamiltonians we are considering, the existence of a connected component with the above stated properties is thus garanteed.

The properties of the flow in $\bar{I}_h^*$ are a consequence of Eqs. (2.4), (2.5), and (2.7). Suppose that there exisits an equilibrium point (r, θ, v, u) of (2.4) in $\bar{I}_h^*$ such that $r \neq 0$. Then we would have $(df/dr)(r) = 0$, $h = f(r)\, V(\theta)$, which contradicts the transversality hypothesis. Therefore, the only equilibrium points of (2.4) in $\bar{I}_h^*$ belong to Λ and their expression is obtained from (2.7) and the equation of $V(\theta)$. ▮

Now we shall study the qualitative behaviour of the global flow on Λ.

Consider the equilibrium points given by Proposition 2.1. A straightforward calculation shows that the dimensions of the stable and unstable manifolds associated to these equilibrium points are as stated in Table 2.1. Since $V''(\pi/2) = -\alpha(1-\mu)$, the sinks and sources on Λ have characteristic exponents with imaginary part different from zero iff $\mu > 1 + (\alpha - 2)^2/8\alpha = \mu^*$. That is, we have spiral sinks and spiral sources on Λ iff $\mu > \mu^*$.

Let us introduce some notation. We shall denote by $B^{u,s}_{+,-}(p, \mu)$ the branch of the unstable (u) or stable (s) invariant manifold associated to the equilibrium point p on Λ contained in $\{u > 0\}(+)$ or in $\{u < 0\}(-)$ in a neighbourhood of p. We shall denote by $P^{u,s}_{+,-}(p, \mu)$ the points where these branches meet the annulus $\{v = 0\}$; that is, $P^{u,s}_{+,-}(p, \mu) = B^{u,s}_{+,-}(p, \mu) \cap \{v = 0\}$. Since it is always $v' \leqslant 0$ (see Eqs. (2.7)), the points $P^{u,s}_{+,-}(p, \mu)$ are well defined.

Let $\Delta\theta(\alpha, \mu)$ be given by $\theta(P^u_+(p^+(0), \mu))$, where $\theta(q)$ denotes the θ-coordinate of the point q.

LEMMA 2.2. *For every* $\alpha \in (2, +\infty)$ *and* $\mu \in [1, +\infty)$,

$$\frac{2 \arcsin \mu^{-\alpha/4}}{\alpha - 2} \leqslant \Delta\theta(\alpha, \mu) \leqslant \frac{\pi}{\alpha - 2}.$$

Table 2.1

Equilibrium points	dim of W^u	dim of W^s	Type in Λ
$p^+(0)$	2	1	Saddle
$p^+(\pi)$	2	1	Saddle
$p^+(\pi/2)$	3	0	Source
$p^+(3\pi/2)$	3	0	Source
$p^-(0)$	1	2	Saddle
$p^-(\pi)$	1	2	Saddle
$p^-(\pi/2)$	0	3	Sink
$p^-(3\pi/2)$	0	3	Sink

Proof. From Eqs. (2.6) and (2.7) we obtain

$$\Delta\theta(\alpha, \mu) = \int_{2^{1/2}\mu^{-\alpha/4}}^{0} \frac{d\theta}{dv}\, dv = \frac{2}{\alpha - 2} \int_{0}^{2^{1/2}\mu^{-\alpha/4}} \frac{dv}{(-2V(\theta) - v^2)^{1/2}}.$$

Since $2\mu^{-\alpha/2} \leqslant -2V(\theta) \leqslant 2$ we get

$$\frac{2}{\alpha - 2} \int_{0}^{2^{1/2}\mu^{-\alpha/4}} \frac{dv}{(2 - v^2)^{1/2}} \leqslant \Delta\theta(\alpha, \mu) \leqslant \frac{2}{\alpha - 2} \int_{0}^{2^{1/2}\mu^{-\alpha/4}} \frac{dv}{(2\mu^{-\alpha/2} - v^2)^{1/2}}$$

and the result follows by integration. ∎

LEMMA 2.3. *For $\alpha \in (2, +\infty)$ fixed, $\Delta\theta(\alpha, \mu)$ is a strictly decreasing function of μ.*

Proof. Introducing on A the angular coordinates (θ, ψ) given by $u = \sqrt{-2V(\theta)} \cos \psi$, $v = \sqrt{-2V(\theta)} \sin \psi$, Eqs. (2.7) become

$$\begin{aligned} \theta' &= \sqrt{-2V(\theta)} \cos \psi, \\ \psi' &= \left(1 - \frac{\alpha}{2}\right) \sqrt{-2V(\theta)} \cos \psi + \frac{(dV/d\theta)(\theta)}{\sqrt{-2V(\theta)}} \sin \psi. \end{aligned} \tag{2.8}$$

Computing the linear approximation of (2.8) in a neighbourhood of the equilibrium point $p^+(0)$, the slope $s^+(\alpha, \mu)$ of $B^u_+(p^+(0), \mu)$ is given by

$$s^+(\alpha, \mu) = \frac{1}{2}\left(1 - \frac{\alpha}{2}\right) - \sqrt{\frac{1}{4}\left(\frac{\alpha}{2} - 1\right)^2 + \frac{\alpha(\mu - 1)}{2\mu}}.$$

Clearly, for α fixed, $s^+(\alpha, \mu)$ is a strictly decreasing function of μ and $s^+(\alpha, (1, +\infty)) = (-\alpha/2, 1 - \alpha/2)$.

Let $\mu_1 < \mu_2$ and suppose $\Delta\theta(\alpha, \mu_1) \leqslant \Delta\theta(\alpha, \mu_2)$. Then there must be a point $(\bar\theta, \bar\psi)$, with $\bar\psi \in (0, \pi/2)$, such that at $(\bar\theta, \bar\psi)$ verifies $(d\psi/d\theta)(\mu_1) < (d\psi/d\theta)(\mu_2)$, where $d\psi/d\theta$ is given by Eqs.(2.8). Therefore we would have,

$$\frac{(1 - \mu_1) \cos \theta \sin \theta}{\mu_1 \cos^2 \theta + \sin^2 \theta} < \frac{(1 - \mu_2) \cos \theta \sin \theta}{\mu_2 \cos^2 \theta + \sin^2 \theta} \tag{2.9}$$

for $\theta = \bar\theta$. But (2.9) implies $\mu_2 < \mu_1$, which is a contradiction. ∎

The next proposition is a direct consequence of Lemmas 2.2 and 2.3.

PROPOSITION 2.4. *The flow on A verifies the following properties (see Fig. 2.2):*

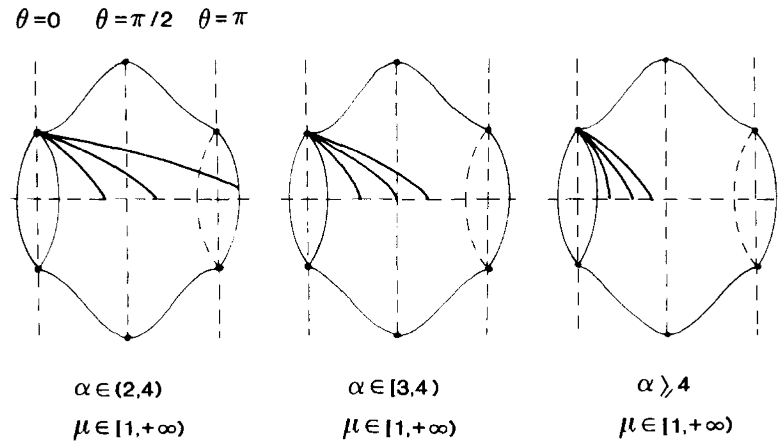

FIG. 2.2. Qualitative features of $B^{\mathrm{u}}_{+}(p^{+}(0), \mu)$.

(a) *Given* $\theta_0 \in [0, 2\pi]$ *and* $k \in \mathfrak{N}\backslash\{0\}$, *there exists* $\alpha \in (2, 4)$ *and* $\mu \in (1, +\infty)$ *such that* $\theta(\alpha, \mu) = \theta_0 + 2k\pi$.

(b) *If* $\alpha \in [3, 4)$ *and* $\mu \in (1, +\infty)$ *then* $0 < \Delta\theta(\alpha, \mu) < \pi$.

(c) *If* $\alpha \in [4, +\infty)$ *and* $\mu \in (1, +\infty)$ *then* $0 < \Delta\theta(\alpha, \mu) < \pi/2$.

Let us define $S_i\colon \mathfrak{R}^+ \times S^1 \times \mathfrak{R}^2 \times \mathfrak{R} \rightleftarrows$ for $i \in \{0, 1, 2\}$ as $S_0(r, \theta, v, u, \tau) = (r, \theta, -v, -u, -\tau)$, $S_1(r, \theta, v, u, \tau) = (r, -\theta, -v, u, -\tau)$, and $S_2(r, \theta, v, u, \tau) = (r, \pi - \theta, -v, u, -\tau)$. Clearly, for $i \in \{0, 1, 2\}$, $S_i \circ S_i = Id$ and S_i leaves system (2.4) invariant. Then we have the following:

PROPOSITION 2.5. *For every* $i \in \{0, 1, 2\}$, S_i *is a symmetry of system* (2.4).

Note that new symmetries may be obtained by composition of the S_i, $i \in \{0, 1, 2\}$. In particular, $S_3(r, \theta, v, u, \tau) = S_2 \circ S_1(r, \theta, v, u, \tau) = (r, \pi + \theta, v, u, \tau)$ is also a symmetry of system (2.4).

LEMMA 2.6. *For* $\alpha \in (2, +\infty)$ *and* $\mu \in (1, +\infty)$, *the qualitative behaviour of* $B^{\mathrm{u}}_{+}(p^{+}(0), \mu)$ *between the points* $p^{+}(0)$ *and* $P^{\mathrm{u}}_{+}(p^{+}(0), \mu)$ *completely determines the qualitative features of the flow on* Λ.

Proof. Since it is always $v' \leqslant 0$ (see Eqs. (2.7)), the qualitative behaviour of $B^{\mathrm{s}}_{+}(p^{+}(0), \mu)$, $B^{\mathrm{s}}_{-}(p^{+}(0), \mu)$, $B^{\mathrm{s}}_{+}(p^{+}(\pi), \mu)$, $B^{\mathrm{s}}_{-}(p^{+}(\pi), \mu)$, $B^{\mathrm{u}}_{+}(p^{-}(0), \mu)$, $B^{\mathrm{u}}_{-}(p^{-}(0), \mu)$, $B^{\mathrm{u}}_{+}(p^{-}(\pi), \mu)$, and $B^{\mathrm{u}}_{-}(p^{-}(\pi), \mu)$ must be as shown in Fig. 2.3a.

Now, suppose that the behaviour of $B^{\mathrm{u}}_{+}(p^{+}(0), \mu)$ between the point $p^{+}(0)$ and the point $P^{\mathrm{u}}_{+}(p^{+}(0), \mu)$ is known. Then, by using the

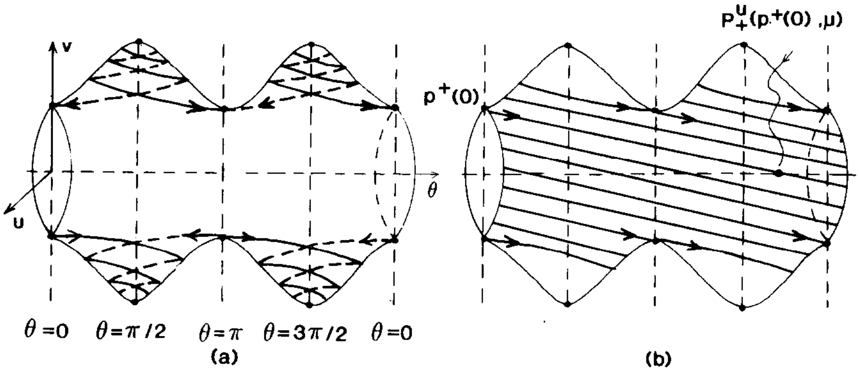

FIG. 2.3. Global flow on Λ. A solution on Λ drawn in a continuous (resp. discontinuous) line means that it is contained in $\Lambda \cap \{u \geqslant 0\}$ (resp. $\Lambda \cap \{u \leqslant 0\}$).

symmetries $S_0 \circ S_1$, S_1, and S_0, we obtain $B^u_-(p^+(0), \mu)$, $B^s_+(p^-(0), \mu)$, and $B^s_-(p^-(0), \mu)$ between the points $p^+(0)$ and $P^u_-(p^+(0), \mu)$, $p^-(0)$ and $P^s_+(p^-(0), \mu)$, $p^-(0)$ and $P^s_-(p^-(0), \mu)$, repectively. Moreover, applying S_3 to the four invariant branches studied until now we obtain the corresponding qualitative behaviour of $B^u_+(p^+(\pi), \mu)$, $B^u_-(p^+(\pi), \mu)$, $B^s_+(p^-(\pi), \mu)$ and $B^s_-(p^-(\pi), \mu)$.
Using once again the fact that $v(\tau)$ is non increasing the global behaviour of the sixteen invariant branches associated to the saddle points of Λ is found and, therefore, the qualitative features of the flow on Λ. ∎

In Fig. 2.3b we have represented the flow on Λ for the case when $\Delta\theta(\alpha, \mu) \in (3\pi/2, 2\pi)$.

Let $\alpha \in (2, +\infty)$ and $\mu > 1$. If $B^u_+(p^+(0), \mu)$ meets the circle $\{v = 0\}$ at $\theta = \theta_0$, $\theta_0 \in [0, 2\pi]$, after turning k times around Λ, $k \in N \cup \{0\}$, we shall denote by $F(\theta_0, k)$ the corresponding flow on Λ.

With this notation, the following theorem comes as a direct consequence of Proposition 2.4 and Lemma 2.6.

THEOREM 2.7. (a) *Given* $\theta_0 \in [0, 2\pi]$ *and* $k \in \aleph \cup \{0\}$, *there exist* $\alpha \in (2, +\infty)$ *and* $\mu \in (1, +\infty)$ *such that the flow on* Λ *is of type* $F(\theta_0, k)$.

(b) *If* $\alpha \in [3, 4)$ *and* $\mu \in (1, +\infty)$, *the flow on* Λ *is of type* $F(\theta_0, 0)$ *with* $\theta_0 \in (0, \pi]$.

(c) *If* $\alpha \in [4, +\infty)$ *and* $\mu \in (1, +\infty)$, *the flow on* Λ *is of type* $F(\theta_0, 0)$ *with* $\theta_0 \in (0, \pi/2]$.

Actually, given $k \in \aleph \cup \{0\}$, there are eight different possibilities for the qualitative behaviour of $F(\theta_0, k)$, according to whether $\theta_0 \in (0, \pi/2)$, $\theta_0 = \pi/2$, $\theta_0 \in (\pi/2, \pi)$, $\theta_0 = \pi$, $\theta_0 \in (\pi, 3\pi/2)$, $\theta_0 = 3\pi/2$, $\theta_0 \in (3\pi/2, 2\pi)$, or $\theta_0 = 2\pi$, that we shall denote by $Fi(k)$, $i = 1, 2, ..., 8$, respectively. Note that the cases $Fi(k)$, $i = 2, 4, 6, 8$ are not generic; anyway, we shall include them in our posterior study because they will not suppose an extra effort.

Equations (2.4) provide us analytical extension of the flow of system (2.1) from I_h^* to $\bar{I}_h^* = I_h^* \cup \Lambda$ and Theorem 2.7 describes, in terms of the parameters $\alpha \in (2, +\infty)$ and $\mu \in (1, +\infty)$, the global behaviour of the flow on Λ. In the next section, we shall see that this extension, along with the study of the invariant manifolds associated to equilibrium points on Λ, brings us new information about the flow on I_h^*.

3. Dynamics on the Negative Energy Levels

In this section we shall always be studying the flow for negative energy. Therefore the compactified connected component $\bar{I}_h^*$ of the energy level I_h (see Proposition 2.1) will be $\bar{I}_h^* = I_h^* \cup \Lambda$, and the flow on $\bar{I}_h^*$ will be given by Eqs. (2.4) of Section 2. From now on, we shall describe the flow of the system we are studying in terms of McGehee's variables (r, θ, v, u, τ), introduced in (2.3).

Consider the annulus S defined by $S = \{(r, \theta, v, u): v = 0,\ \frac{1}{2}(u^2 + v^2) + r^\alpha f(r)\, V(\theta) = r^\alpha h\} = \bar{I}_h^* \cap \{v = 0\}$.

Proposition 3.1. *For h small enough, the flow of Eqs.* (2.4) *is transversal to S, and all the orbits starting at a point of S tend to an equilibrium point on Λ when $\tau \to +\infty$.*

Proof. From Eqs. (2.4) and (2.5), v' is given on S by $r^\alpha(2h - 2f(r)\, V(\theta) - rf'(r)\, V(\theta))$, which tends to $(\alpha - 2)\, V(\theta)$ when $r \to 0$. So, $v' < 0$ in S provided that r is small enough. But, taking h small enough, the projection of I_h^* on the r-axis is contained in an arbitrarily small neighbourhood of zero, and the result follows. ∎

Notice that for functions $f(r)$ of the form $\sum_x r^{-x}$ Proposition 3.1 holds for every $h < 0$. Likewise, the hypothesis that the energy parameter is small enough that will appear in the following results becomes simply $h < 0$ for functions $f(r)$ of that type.

According to the notation introduced in Section 2, we shall denote by $W^{u}_{+,-}(p, \mu)$ (resp. $W^{s}_{+,-}(p, \mu)$) the unstable (resp. stable) invariant manifold associated to the saddle point $p \in \Lambda$ such that $W^{u}_{+,-}(p, \mu) \cap \Lambda = B^{u}_{+,-}(p, \mu)$ (resp. $W^{s}_{+,-}(p, \mu) \cap \Lambda = B^{s}_{+,-}(p, \mu)$). With this notation, the following proposition follows easy from the definition of the symmetries S_0, S_2, and S_3 (see Proposition 2.5).

PROPOSITION 3.2. *The following relations hold*:

$$W^{\mathrm{u}}_{+}(p^{+}(\pi), \mu) = S_3(W^{\mathrm{u}}_{+}(p^{+}(0), \mu)),$$
$$W^{\mathrm{u}}_{-}(p^{+}(\pi), \mu) = S_2 \circ S_0(W^{\mathrm{u}}_{+}(p^{+}(0), \mu)),$$
$$W^{\mathrm{u}}_{-}(p^{+}(0), \mu) = S_2 \circ S_0(W^{\mathrm{u}}_{+}(p^{+}(\pi), \mu)),$$
$$W^{\mathrm{s}}_{+}(p^{-}(\pi), \mu) = S_2(W^{\mathrm{u}}_{+}(p^{+}(0), \mu),$$
$$W^{\mathrm{s}}_{-}(p^{-}(\pi), \mu) = S_2(W^{\mathrm{u}}_{-}(p^{+}(0), \mu),$$
$$W^{\mathrm{s}}_{+}(p^{-}(0), \mu) = S_2(W^{\mathrm{u}}_{+}(p^{+}(\pi), \mu),$$
$$W^{\mathrm{s}}_{-}(p^{-}(0), \mu) = S_2(W^{\mathrm{u}}_{-}(p^{+}(\pi), \mu).$$

In order to study the flow on I_h^* we shall define on the surface S a partition into closed regions. To each one of these regions will correspond a class of orbits with the same qualitative behaviour. The construction of these regions is based on the behaviour of the intersections of the manifolds $W^{\mathrm{u,s}}_{+,-}(p^{+,-}(\theta_0), \mu)$, $\theta_0 = 0, \pi$, with the surface S.

Denote by $\sigma^{\mathrm{u}}_{+,-}(p, \mu)$ (resp. $\sigma^{\mathrm{s}}_{+,-}(p, \mu)$) the first intersection of $W^{\mathrm{u}}_{+,-}(p, \mu)$ (resp. $W^{\mathrm{s}}_{+,-}(p, \mu)$) in forward time (resp. backward time), where p denotes one of the saddle points on Λ (see Table 2.1). Recall also that $P^{\mathrm{u,s}}_{+,-}(p, \mu)$ denotes the intersection $B^{\mathrm{u,s}}_{+,-}(p, \mu) \cap \{v = 0\}$.

THEOREM 3.3. *For h small enough, the endpoints of the curves $\sigma^{\mathrm{u,s}}_{+,-}(p, \mu)$ are $P^{\mathrm{u,s}}_{+,-}(p, \mu)$ and the projections $\pi(p) = (\theta = \theta_0, u = 0)$ of p on the annulus $\{v = 0\}$. Moreover, $\sigma^{\mathrm{u,s}}_{+,-}(p, \mu) \backslash \pi(p)$ is contained either in $\{u > 0\}$ or in $\{u < 0\}$, and the θ-coordinate of its points is maximal at $P^{\mathrm{u,s}}_{+,-}(p, \mu)$ (see Fig. 3.1).*

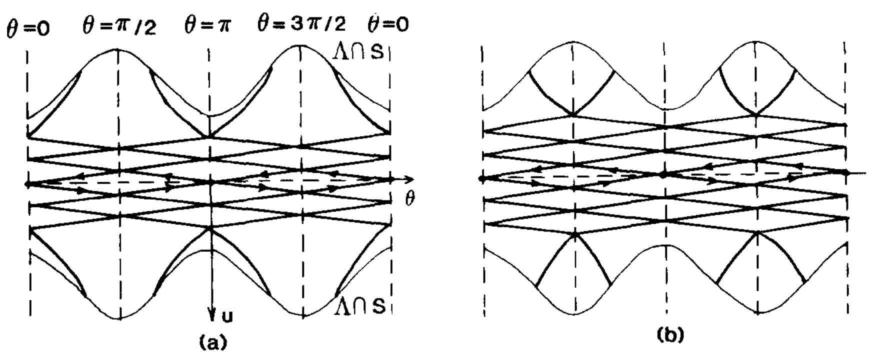

FIG. 3.1. Curves $\sigma^{\mathrm{u,s}}_{+,-}$. (a) Case $F1(1)$. (b) Case $F3(1)$.

In order to prove this theorem we need three previous lemmas.

LEMMA 3.4. *For h small enough and for every $\theta_0 \in \{0, \pi/2, \pi, 3\pi/2\}$ there exists a homothetic heteroclinic orbit $\gamma_h(\theta_0)$ whose α-limit (resp. ω-limit) set is the point $p^+(\theta_0)$ (resp. $p^-(\theta_0)$).*

Proof. Consider $\gamma_h(\theta_0) = \{(r, \theta, v, u): \theta = \theta_0, \ u = 0, \ r^\alpha h = v^2/2 + r^\alpha f(r) V(\theta_0)\}$. Clearly, $\gamma_h(\theta_0)$ satisfies Eqs. (2.4) and (2.5). Moreover, from Proposition 3.1, $\gamma_h(\theta_0) \to p^+(\theta_0)$ (resp. $p^-(\theta_0)$) when $\tau \to -\infty$ (resp. $\tau \to +\infty$). ∎

LEMMA 3.5. *Let h be such that Lemma 3.4 holds and let $\gamma_h(\theta_0)$ be the heteroclinic orbit given by this lemma with $\theta_0 \in \{0, \pi\}$, Consider the parametrization of $\gamma_h(\theta_0)$ by τ such that $v(\tau = 0) = 0$, and let $q(\tau) = (r(\tau), \theta_0, v(\tau), 0)$ be a point of $\gamma_h(\theta_0)$ and $\mathbf{q}(\tau)$ the unit tangent vector to $\gamma_h(\theta_0)$ at $q(\tau)$. Then, the tangent plane to $W^{\mathrm{u}}(p^+(\theta_0))$ at $q(\tau)$ is generated by $\mathbf{q}(\tau)$ and $\mathbf{t}(\tau)$, where $\mathbf{t}(\tau)$ is of the form $(0, \cos\phi(\tau), 0, \sin\phi(\tau))$.*

Proof. Let $T_{q(\tau)}\bar{I}_h^*$ (resp. $T_{q(\tau)}W^{\mathrm{u}}(p^+(\theta_0))$) denote the tangent space to $\bar{I}_h^*$ (resp. $W^{\mathrm{u}}(p^+(\theta_0))$) at $q(\tau)$. Clearly, $T_{q(\tau)}W^{\mathrm{u}}(p^+(\theta_0))$ is a two-dimensional subspace of the three-dimensional space $T_{q(\tau)}\bar{I}_h^*$, and $\mathbf{q}(\tau) \in T_{q(\tau)}W^{\mathrm{u}}(p^+(\theta_0))$.

Consider the plane $\mathscr{P}(\theta_0) = \{(r, \theta, v, u): \theta = \theta_0, u = 0\}$. The vector $\mathbf{q}(\tau)$ belongs also to the tangent space to $\mathscr{P}(\theta_0)$ at $q(\tau)$. On the other hand, from Eqs. (2.5), $\mathscr{P}(\theta_0)$ intersects $\bar{I}_h^*$ transversally. Therefore, $T_{q(\tau)}\bar{I}_h^*$ is generated by the vectors $\mathbf{q}(\tau)$, $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$. ∎

LEMMA 3.6. *Let $\phi(\tau)$ be the angle given by Lemma 3.5. Then, $\phi(\tau) \in (0, \pi/2)$ for every $\tau \in (-\infty, 0]$.*

Proof. Denoting by $\phi(-\infty)$ the limit $\lim_{\tau \to -\infty} \phi(\tau)$ we have,

$\phi(-\infty) = \arctan \lambda,$

$$\lambda = -\left(2 - \frac{\alpha}{4}\right) A + \left(\left(2 - \frac{\alpha}{4}\right)^2 A^2 - V''(0)\right)^{1/2}, \qquad A = 2^{1/2} \mu^{-\alpha/4},$$

and so $\phi(-\infty) \in (0, \pi/2)$.

The differential equation verified by $\phi(\tau)$ is obtained from the normal variational equations along the orbit $\gamma_h(\theta_0)$. From Eqs. (2.4) and the proof of Lemma 3.5, these variational equations restricted to $\bar{I}_h^*$ correspond to the matrix

$$\begin{pmatrix} 0 & 1 \\ -r(\tau)^\alpha f(r(\tau)) \dfrac{d^2 V}{d\theta^2}(0) & \dfrac{(\alpha - 2)}{2} v(\tau) \end{pmatrix},$$

where $r(\tau)$, $v(\tau)$ is the parametrization of $\gamma_h(0)$ by the time τ such that $v(\tau=0)=0$. Introducing polar coordinates ρ, φ in the (θ, u) plane through $\theta=\rho\cos\varphi$, $u=\rho\sin\varphi$, we get

$$\varphi' = a(\tau)\cos^2\varphi - \sin^2\varphi + b(\tau)\sin\varphi\cos\varphi, \tag{3.1}$$

where $a(\tau) = -r(\tau)^\alpha f(r(\tau))(d^2V/d\theta^2)(0)$ and $b(\tau) = ((\alpha-2)/2)v(\tau)$.

Clearly, $\phi(\tau)$ satisfies (3.1) with initial condition $\phi(-\infty) = \arctan\lambda$.

Consider the function $f_1(\tau) = a(\tau)\cos^2\varphi$, $f_2(\tau) = -\sin^2\varphi + b(\tau)\sin\varphi\cos\varphi$. For τ small enough, $a(\tau)$ is close to $-V''(0)$ and it is easy to check that for every $\tau\in(-\infty, 0)$ there exists a unique $\phi^*(\tau)\in(0, \pi/2)$ such that $\varphi'(\tau, \phi^*(\tau))=0$ and $\varphi'(\tau,\varphi)>0$ (resp. <0) for $\varphi\in(0,\phi^*(\tau))$ (resp. $\varphi\in(\phi^*(\tau),\pi/2)$); moreover, from Eqs. (3.1), we obtain that $\phi^*(\tau)$ is a monotonically decreasing function of τ and $\lim_{\tau\to-\infty}\phi^*(\tau) = \arctan\lambda = \phi(-\infty)$. Hence, there exists τ_0 such that $\phi'(\tau)<0$ for every $\tau\in(-\infty,\tau_0)$. We claim that $\phi(\tau)$ is monotonically decreasing in $(-\infty, 0]$. Then, the result follows since $0<\phi^*(\tau)<\phi(\tau)<\pi/2$ for every $\tau\in(-\infty, 0]$. To prove the claim, suppose there exists $\tau_1>\tau_0$ such that $\phi'(\tau_1)=0$ and $\phi(\tau_1+\varepsilon)>0$ for ε small enough. Then, $\phi(\tau_1+\varepsilon)>\phi(\tau_1)=\phi^*(\tau_1)>\phi^*(\tau_1+\varepsilon)$, which implies $\phi'(\tau_1+\varepsilon)<0$, a contradiction. ∎

Proof of Theorem 3.3. We shall study $\sigma^{\mathrm{u}}_{+}(p^+(0),\mu)$ and show that it has the claimed properties (using the symmetries of Proposition 3.2, the result follows also for the remaining curves).

Let $\bar\sigma^{\mathrm{u}}_{+}(s)$, $s\in[0,+\infty]$, be an arc contained in $W^{\mathrm{u}}_{+}(p^+(0),\mu)$ such that $\bar\sigma^{\mathrm{u}}_{+}(0)\in\gamma_h(0)$ and $\bar\sigma^{\mathrm{u}}_{+}(+\infty)\in B^{\mathrm{u}}_{+}(p^+(0),\mu)$, see Fig. 3.2. The arc $\bar\sigma^{\mathrm{u}}_{+}(s)$ gives us a natural parametrization $\sigma^{\mathrm{u}}_{+}(s)$ of $\sigma^{\mathrm{u}}_{+}(p^+(0),\mu)$, with the parameter $s\in[0,+\infty]$, and such that $\sigma^{\mathrm{u}}_{+}(0)=(0,0)\in S$,

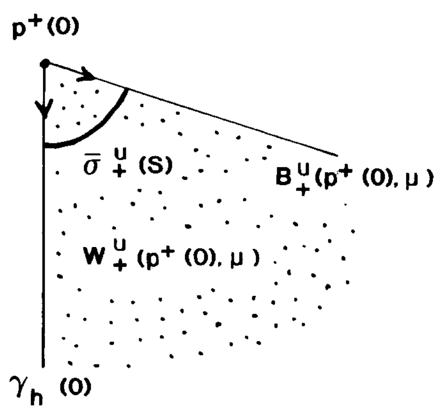

FIG. 3.2. The arc $\bar\sigma^{\mathrm{u}}_{+}(s)$, $s\in[0,+\infty]$.

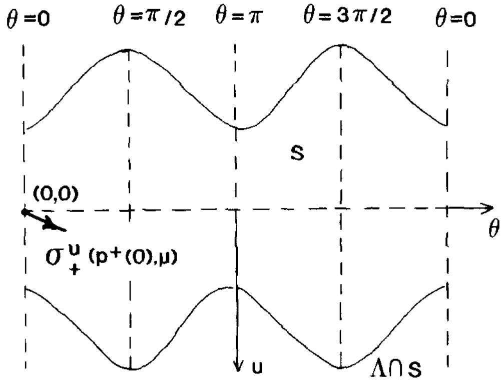

FIG. 3.3. Local behaviour of $\sigma^{u}_{+}(p^{+}(0), \mu)$.

$\lim_{s \to +\infty} \sigma^{u}_{+}(s) = (u(+\infty), \theta(+\infty))$, where $u(+\infty) \in S \cap \{u > 0\} \cap \Lambda$ and $\theta(+\infty) \in ((i-1)\pi/4, (i+1)\pi/4)$ if the flow is of type $Fi(k)$ with i odd, or $\theta(+\infty) = i\pi/4$ if the flow is of type $Fi(k)$ with i even.

From Lemma 3.6, we know that the tangent vector to $W^{u}_{+}(p^{+}(0), \mu)$ at $\sigma^{u}_{+}(0)$ belongs to $(0, \pi/2)$. Therefore, the behaviour of $\sigma^{u}_{+}(p^{+}(0), \mu)$ in a neighbourhood of $(0, 0)$ is as shown in Fig. 3.3.

Now we shall prove that the curve $\sigma^{u}_{+}(s)$, $s \in (0, +\infty)$, cannot intersect the circle $S \cap \{u = 0\}$ and, furthermore, denoting by $\theta(s)$ the θ-coordinate of the point $\sigma^{u}_{+}(s)$, $s \in [0, +\infty]$, we have $\theta(+\infty) > \theta(s)$, $s \in [0, +\infty)$. This will finish the proof of Theorem 3.3.

Suppose there exists $\bar{\theta}, \bar{v}, \bar{u}$, such that the solution of (2.4) with these initial conditions verifies $\theta(\bar{\tau}) \in (\pi/2, \pi)$, $v(\bar{\tau}) = 0$ and $u(\bar{\tau}) = 0$, for a certain value $\bar{\tau}$ of τ. We consider the case $u(\tau) > 0$ for $\tau < \bar{\tau}$. Otherwise one can reduce to this case. Then we have Fig. 3.4 with v as in Fig. 3.5.

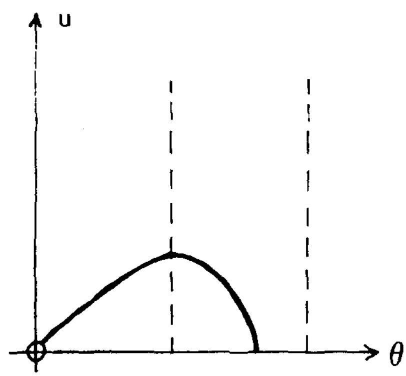

FIG. 3.4.

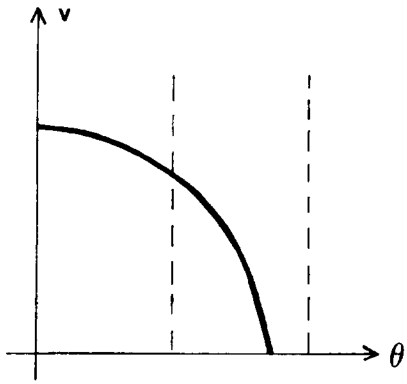

FIG. 3.5.

From Eqs. (2.4) we obtain

$$\frac{du^2}{d\theta}=2\left(\frac{(\alpha-2)}{2}uv-r^\alpha f(r)\frac{dV}{d\theta}(\theta)\right)$$

and so

$$\int_{\bar{\theta}}^{\theta(\bar{\tau})}\frac{du^2}{d\theta}\,d\theta+\int_{\bar{\theta}}^{\theta(\bar{\tau})}2r^\alpha f(r)\;V'(\theta)\,d\theta=(\alpha-2)\int_{\bar{\theta}}^{\theta(\bar{\tau})}uv\,d\theta. \tag{3.2}$$

Integrating the first two terms of (3.2) yields

$$-\bar{u}^2+2(r^*)^\alpha f(r^*)(V(\theta(\bar{\tau}))-V(\bar{\theta}))=(\alpha-2)\int_{\bar{\theta}}^{\theta(\bar{\tau})}uv\,d\theta,$$

where $r^*=r(\tau^*)$, $\tau^*\in(0,\bar{\tau})$. From (3.2), we have

$$V(\theta(\bar{\tau}))-V(\bar{\theta})\geqslant\left(\frac{\alpha}{2}-1\right)\frac{1}{(r^*)^\alpha f(r^*)}\int_{\bar{\theta}}^{\theta(\bar{\tau})}uv\,d\theta,$$

or, since $V(\theta(\bar{\tau}))-V(\bar{\theta})\leqslant V(\pi)-V(\bar{\theta})=V(0)-V(\bar{\theta})$,

$$V(0)-V(\bar{\theta})\geqslant\left(\frac{\alpha}{2}-1\right)\frac{1}{(r^*)^\alpha f(r^*)}\int_{\bar{\theta}}^{\theta(\bar{\tau})}uv\,d\theta. \tag{3.3}$$

Now denote by v_+ the value of v on S^+ and let θ_1 be such that $v(\theta_1)=v_+/2$ if $v(\pi/2)<v_+/2$, $\theta_1=\pi/2$ otherwise. Then

$$\int_{\bar{\theta}}^{\theta(\bar{\tau})}uv\,d\theta>\int_{\bar{\theta}}^{\theta_1}uv\,d\theta>\frac{v_+}{2}\int_{\bar{\theta}}^{\theta_1}u\,d\theta, \tag{3.4}$$

because for h small enough $dv/d\theta < 0$ (see Eqs. (2.4)). Using again Eqs. (2.4), we have

$$u(\theta) - \bar{u} = \int_{\bar{\theta}}^{\theta} \frac{du}{d\theta}\, d\theta > \left(\frac{\alpha}{2} - 1\right) \frac{v_+}{2} (\theta - \bar{\theta})$$

and so

$$\int_{\bar{\theta}}^{\theta} u\, d\theta > \int_{\bar{\theta}}^{\theta} \left(\frac{\alpha}{2} - 1\right) \frac{v_+}{2} (\theta - \bar{\theta})\, d\theta. \tag{3.5}$$

Introducing (3.5) in (3.4), we obtain

$$\int_{\bar{\theta}}^{\theta(\bar{\tau})} uv\, d\theta > \left(\frac{\alpha}{2} - 1\right) \frac{v_+^2}{4} \int_{\bar{\theta}}^{\theta_1} (\theta - \bar{\theta})\, d\theta$$

and (3.3) becomes

$$V(0) - V(\bar{\theta}) > \frac{((\alpha/2) - 1)^2\, v_+^2}{8(r^*)^\alpha f(r^*)} (\theta_1 - \bar{\theta})^2. \tag{3.6}$$

Now, taking $(\bar{\theta}, \bar{v}, \bar{u})$ sufficiently close to $p^+(0)$, the first member of (3.6) may be made arbitrarily small. But the second member has a lower bound, because θ_1 cannot be arbitrarily close to $\bar{\theta}$. So, we have a contradiction, and the result follows. ∎

Before proceeding let us introduce some notation. We shall say that an orbit $p(\tau) = (r(\tau), \theta(\tau), v(\tau), u(\tau))$ is an upper positive (resp. upper negative) ejection, and we shall denote it by $e(+, \mathrm{u})$ (resp. $e(-, \mathrm{u})$), if $p(\tau) \in W^{\mathrm{u}}_-(p^+(\pi), \mu)$ (resp. $p(\tau) \in W^{\mathrm{u}}_+(p^+(0), \mu)$). We shall say that an orbit $p(\tau)$ is an upper positive (resp. upper negative) collision, and we shall denote it by $c(+, \mathrm{u})$ (resp. $c(-, \mathrm{u})$), if $p(\tau) \in W^{\mathrm{s}}_-(p^-(0), \mu)$ (resp. $p(\tau) \in W^{\mathrm{s}}_+(p^-(\pi), \mu)$). A lower positive or negative ejection or collision will be defined by applying symmetry S_3 to the corresponding upper orbit, and we shall denote them by $e(+, \mathrm{l})$, $e(-, \mathrm{l})$, $c(+, \mathrm{l})$ and $c(-, \mathrm{l})$, respectively. These orbits are represented in Fig. 3.6, which may be obtained from Fig. 2.3b and Lemma 3.4.

Let N: $S \backslash \{(\pi/2, 0), (3\pi/2, 0)\} \to \Re$ be the function that maps a point p in its domain into the number of times, $N(p)$, that the orbit through p crosses the axis $\theta = \pi/2, 3\pi/2$, when $v(\tau)$ goes from $-V(0)$ to $V(0)$ as τ increases. In computing $N(p)$, an ejection before p or a collision after p must also be considered as crossings.

From equations (2.4) and (2.5) we have $v' = (1 - \alpha/2)u^2 + \alpha r^\alpha h - r^\alpha V(\theta)(\alpha f(r) + r(df/dr)(r))$. Thus, for $\alpha > 2$ and h small enough, we have $v' < 0$ out of Λ. Since the equilibrium points on Λ correspond to

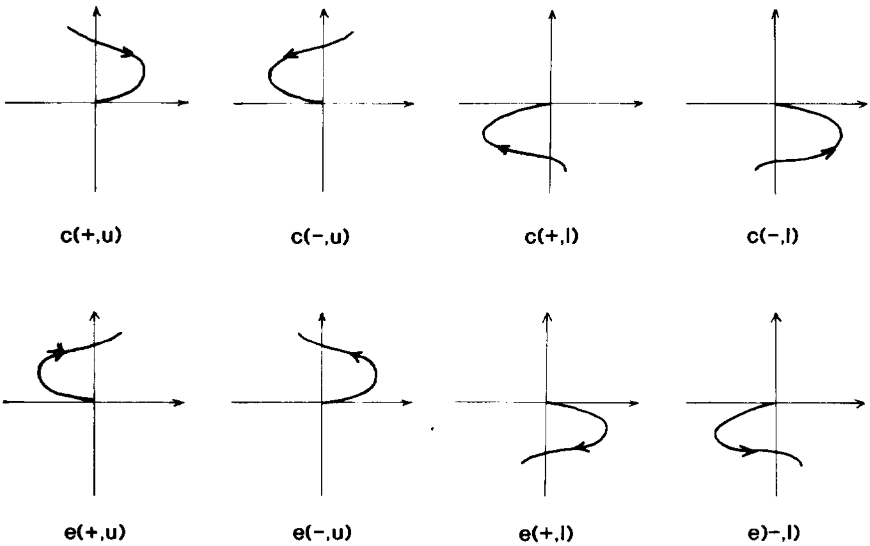

FIG. 3.6. The different types of ejections and collisions.

$v = -V(0)$, $-V(\pi/2)$, $V(0)$, $V(\pi/2)$, the v coordinate of an orbit varies at least from $-V(0)$ to $V(0)$. Note also that the orbit through p cannot be tangent to the axis $\theta = \pi/2, 3\pi/2$, since otherwise it would coincide with the homothetic orbit at $\theta = \pi/2$ or $\theta = 3\pi/2$. Thus, $N(p)$ is well defined.

PROPOSITION 3.7. *Consider the partition of* $S \backslash \{(\pi/2, 0), (3\pi/2, 0)\}$ *defined by the curves* $\sigma^{u,s}_{+,-}(p, \mu)$.

(a) *If p belongs to one of the regions that surround* $(\pi/2, 0)$, $(3\pi/2, 0)$, *then* $N(p) = 2$.

(b) *If p belongs to one of the regions that surround the $\theta = 0$ and $\theta = \pi$ axis that connect the two regions mentioned in* (a), *then* $N(p) = 3$.

(c) *$N(p)$ increases one unity as we cross the boundary between two regions of the partition in the sense of decreasing distance to the collision manifold* (*see Fig.* 3.8).

The proposition follows immediately from the next three lemmas.

LEMMA 3.8. *The following properties hold*:

(a) *N is continuous in* $S \backslash \{\sigma^s_-(p^-(\pi), \mu) \cup \sigma^s_+(p^-(\pi), \mu) \cup \sigma^s_-(p^-(0), \mu) \cup \sigma^s_+(p^-(0), \mu) \cup \sigma^u_+(p^+(\pi), \mu) \cup \sigma^u_-(p^+(\pi) \cup \sigma^u_+(p^+(0), \mu) \cup \sigma^u_-(p^+(0), \mu)\}$.

(b) *In its discontinuities, N either decreases or increases by one unity.*

The proof may be found in [D, pp. 303–304].

LEMMA 3.9. *Given $p \in S$, the following properties hold:*

(a) $\quad p \in (\sigma^{s}_{-}(p^{-}(0), \mu) \cap \sigma^{u}_{-}(p^{+}(0), \mu)) \Rightarrow N(p)$ *even,*

(b) $\quad p \in (\sigma^{s}_{-}(p^{-}(0), \mu) \cap \sigma^{u}_{-}(p^{+}(\pi), \mu)) \Rightarrow N(p)$ *odd,*

(c) $\quad p \in (\sigma^{s}_{+}(p^{-}(\pi), \mu) \cap \sigma^{u}_{+}(p^{+}(0), \mu)) \Rightarrow N(p)$ *odd,*

(d) $\quad p \in (\sigma^{s}_{+}(p^{-}(\pi), \mu) \cap \sigma^{u}_{+}(p^{+}(\pi), \mu)) \Rightarrow N(p)$ *even,*

(e) $\quad p \in (\sigma^{s}_{+}(p^{-}(0), \mu) \cap \sigma^{u}_{+}(p^{+}(\pi), \mu)) \Rightarrow N(p)$ *odd,*

(f) $\quad p \in (\sigma^{s}_{+}(p^{-}(0), \mu) \cap \sigma^{u}_{+}(p^{+}(0), \mu)) \Rightarrow N(p)$ *even,*

(g) $\quad p \in (\sigma^{s}_{-}(p^{-}(\pi), \mu) \cap \sigma^{u}_{-}(p^{+}(\pi), \mu)) \Rightarrow N(p)$ *even,*

(h) $\quad p \in (\sigma^{s}_{-}(p^{-}(\pi), \mu) \cap \sigma^{u}_{-}(p^{+}(0), \mu)) \Rightarrow N(p)$ *odd.*

Proof. The result follows immediately from the asymptotic behaviour of the orbits of $W^{u,s}_{+,-}(p, \mu)$, $p = p^{+,-}(\theta_0)$, $\theta_0 = 0, \pi$, and from the fact that the orbits through p cannot be tangent to the axis $\theta = \pi/2, 3\pi/2$. ∎

COROLLARY 3.10. *If p is in the hypothesis of Lemma* 3.9, *then the values of N in a neighbourhood of p are as shown in Fig.* 3.7.

The results obtained so far provide us with a partition of the curves $\sigma^{u,s}_{+,-}(p^{+,-}(\theta_0))$, $\theta_0 = 0, \pi$, according to the qualitative behaviour of the orbits. For instance, the segment of $\sigma^{s}_{-}(p^{-}(\pi), \mu)$ marked with a thick line in Fig. 3.8a decomposes itself in segments A and B and the point p' defined by the intersection with another curve $\sigma^{s,u}_{+,-}$. To these subsets of $\sigma^{s}_{-}(p^{-}(\pi), \mu)$ correspond the qualitative behaviours represented in Fig. 3.9.

In order to extend this classification to the whole surface S, we shall need more notation.

Let $p(\tau)$ be an orbit such that $p(\tau) \notin (W^{u}_{+}(p^{+}(0), \mu) \cup W^{u}_{-}(p^{+}(0), \mu) \cup W^{u}_{+}(p^{+}(\pi), \mu) \cup W^{u}_{-}(p^{+}(\pi), \mu))$, $p(\tau) \notin (W^{s}_{+}(p^{+}(0), \mu) \cup W^{s}_{-}(p^{+}(0), \mu) \cup W^{s}_{+}(p^{+}(\pi), \mu) \cup W^{s}_{-}(p^{+}(\pi), \mu))$. Since $v' < 0$ out of A, it is either $p(\tau) \to p^{+}(\pi/2)$ or $p(\tau) \to p^{+}(3\pi/2)$ when $\tau \to -\infty$. Furthermore, the qualitative behaviour of $p(\tau)$ for $\tau \in (-\infty, 0)$, where $p(0) \in \{v = +V(0)\}$, is as shown in Fig. 3.10.

Similarly, let $p(\tau)$ be an orbit such that $p(\tau) \notin (W^{u}_{+}(p^{-}(0), \mu) \cup W^{u}_{-}(p^{-}(0), \mu) \cup W^{u}_{+}(p^{-}(\pi), \mu) \cup W^{u}_{-}(p^{-}(\pi), \mu))$, $p(\tau) \notin (W^{s}_{+}(p^{-}(0), \mu) \cup W^{s}_{-}(p^{-}(0), \mu) \cup W^{s}_{+}(p^{-}(\pi), \mu) \cup W^{s}_{-}(p^{-}(\pi), \mu))$. The possible qualitative behaviours for $p(\tau)$, $\tau \in (0, +\infty)$, where $p(0) \in \{v = -V(0)\}$, are as shown in Fig. 3.11.

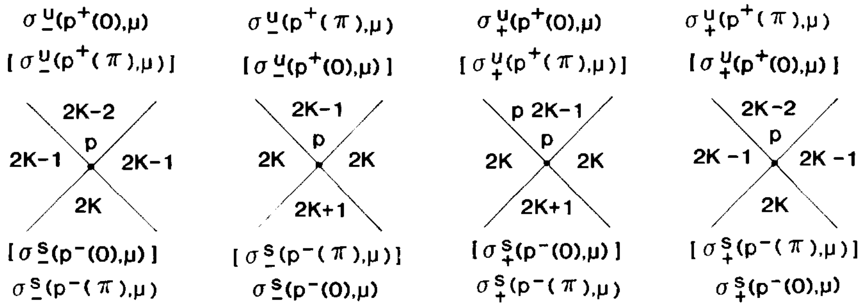

FIG. 3.7. Values of N in a neighbourhood of $p \in \sigma^s \cap \sigma^u$. We use $[\cdot]$ in order to represent another case in the same picture.

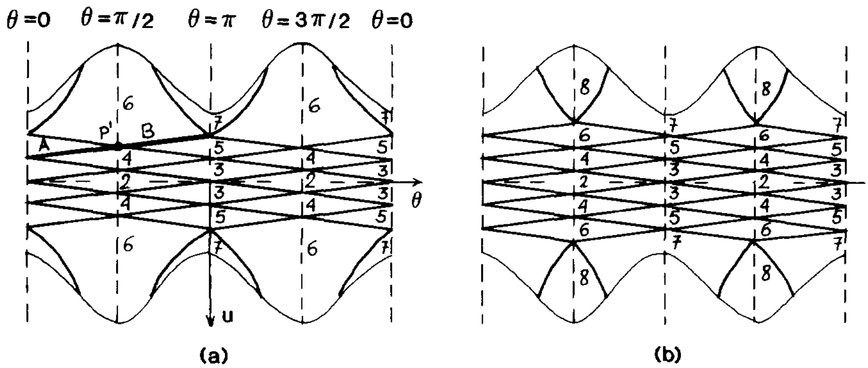

FIG. 3.8. (a) Values of N for a flow of type $F1(1)$. (b) Values of N for a flow of type $F3(1)$.

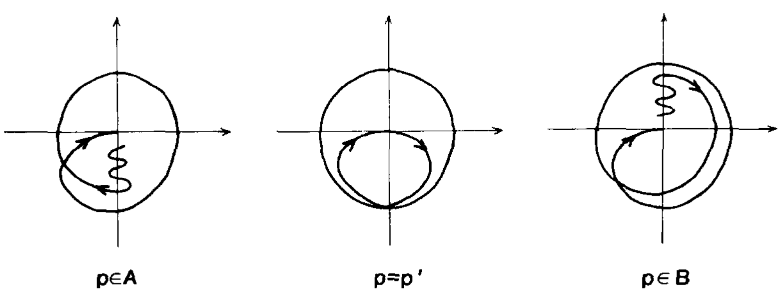

FIG. 3.9. Trajectories corresponding to A, p', and B of Fig. 3.8a.

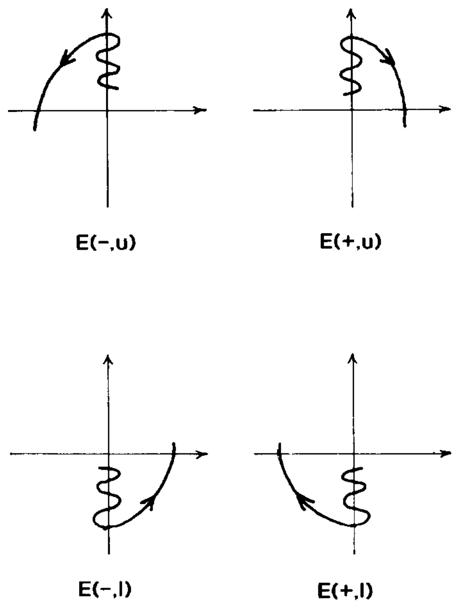

FIG. 3.10. Possible qualitative behaviours for $p(\tau)$, $\tau \in (-\infty, 0)$, when $p(0) \in \{v = V(0)\}$.

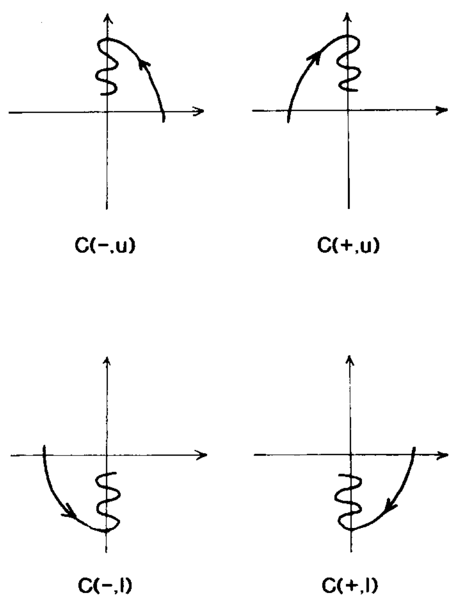

FIG. 3.11. Possible qualitative behaviours for $p(\tau)$, $\tau \in (0, +\infty)$, when $p(0) \in \{v = -V(0)\}$.

In analogy with the notation used for the orbits represented in Fig. 3.6, we shall denote by E(−, u), E(−, l), E(+, u), E(+, l), C(−, u), C(−, l), C(+, u) and C(+, l) the eight assymptotic trajectories of Fig. 3.10 and 3.11. As before, E (resp. C) stands for ejection (resp. collision), − (resp. +) stands for counterclockwise (resp. clockwise) motion, and u (resp. l) stands for upper (resp. lower) halfplane. Notice that for the trajectories of type C(·, ·) or E(·, ·), the number of crossings of the axis $\theta = \pi/2, 3\pi/2$ is finite for $\mu < \mu^*$ and infinite for $\mu > \mu^* = 1 + (\alpha - 2)^2/8\alpha$ (see Section 2).

Let $A \in \{$E(+, u), E(−, u), E(+, l), E(−, l), e(+, u), e(−, u), e(+, l), e(−, l)$\}$, $B \in \{$C(+, u), C(−, u), C(+, l), C(−, l), c(+, u), c(−, u), c(+, l), c(−, l)$\}$ and $n \in \mathfrak{N}$. We shall denote by $[A, n, B]$ the set of points $p \in S$ such that $N(p) = n$, $p(\tau)$ behaves like A when $\tau \to -\infty$ and $p(\tau)$ behaves like B when $\tau \to +\infty$ (see two examples of Fig. 3.12).

Using the results obtained in Theorem 3.3, Proposition 3.7 and Lemma 3.9, together with the fact that $W^{u,s}_{+,-}(p, \mu)$, $p = p^+(0)$, $p^-(0)$, $p^+(\pi)$, $p^-(\pi)$, are invariant for the flow we obtain the following result.

THEOREM 3.11. *Suppose h is small enough, let $\alpha \in (2, +\infty)$ and let $\mu \in (1, +\infty)$ be such that the flow on A is of type $Fi(k)$, $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, $k \in \aleph \cup \{0\}$. The partition on S defined by the curves $\sigma^{u,s}_{+,-}(p, \mu)$, $p = p^{+,-}(\theta_0)$, $\theta_0 = 0, \pi$, has (at least) $2 + 16k + 4[(i+1)/2]$ closed regions R_j (here [] denotes the integer part). This partition gives us a classification of the qualitative behaviour of the solutions of* (2.4) *according to the following rules:*

(a) *If p belongs to the interior of a region R_j then the orbit through p is of type* [E(·, ·), *n*, C(·, ·)].

(b) *If p belongs to the common boundary of two closed regions R_j and R_k, and this boundary is contained on $\sigma^{u}_{+,-}$ (resp. $\sigma^{s}_{+,-}$) then the orbit through p starts (resp. ends) with a motion of type* e(·, ·) *(resp.* c(·, ·)*).*

(c) *If p belongs to the interior of the region centered around the*

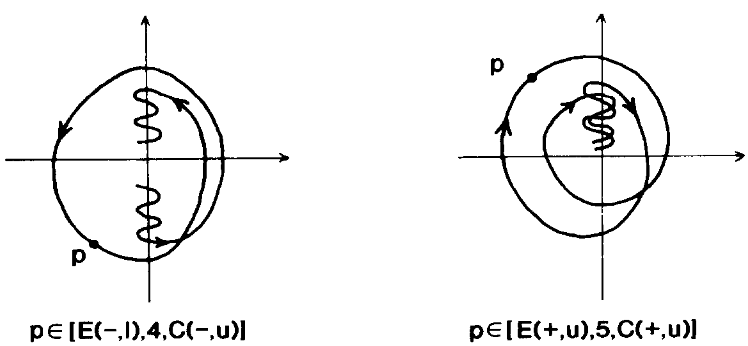

FIG. 3.12. Some examples of orbits of the sets $[A, n, B]$.

point $(\pi/2, 0) \in S$ *(resp.* $(3\pi/2, 0)$*) then the orbit through p is of type* $[\mathrm{E}(\cdot, \mathrm{u}), 2, \mathrm{C}(\cdot, \mathrm{u})]$ *(resp.* $[\mathrm{E}(\cdot, \mathrm{l}), 2, \mathrm{C}(\cdot, \mathrm{l})]$*).*

(d) *If p belongs to the interior centered around of the* $\theta = \pi$ *axis and contained in* $\{u \geqslant 0\}$ *which connects the two regions mentioned in* (c), *then the orbit through p is of type* $[\mathrm{E}(-, \mathrm{u}), 3, \mathrm{C}(-, \mathrm{l})]$.

(e) *If p belongs to a region contained in* $\{u \geqslant 0\}$ *(resp.* $\{u \leqslant 0\}$*), then the orbit through p is of type* $[A(\cdot, -), n, B(\cdot, -)]$ *(resp.* $[A(\cdot, +), n, B(\cdot, +)]$*).*

(f) *If p belongs to a region centered around of the* $\theta = \pi/2$ *or* $\theta = 3\pi/2$ *axis, then the orbit through p is of type* $[A(\mathrm{u}, \cdot), n, B(\mathrm{u}, \cdot)]$, *we shall call it type* u, *or of type* $[A(\mathrm{l}, \cdot), n, B(\mathrm{l}, \cdot)]$, *we shall call it type* l. *Moreover, if the boundaries of two of these regions intersect and one of them is of type* u *(resp. type* l*) the other is of type* l *(resp. type* u*).*

(g) *If p belongs to a region centered around the* $\theta = 0$ *or* $\theta = \pi$ *axis, then the orbit through p is of type* $[A(\mathrm{u}, \cdot), n, B(\mathrm{l}, \cdot)]$, *we shall call it type* u–l, *or type* $[A(\mathrm{l}, \cdot), n, B(\mathrm{u}, \cdot)]$, *we shall call it type* l–u. *Moreover, if the boundaries of two of these regions intersect and one of them is of type* u–l *(resp.* l–u*), the other is of type* l–u *(resp.* u–l*).*

Note that we could have more than $2 + 16k + 4[(i+1)/2]$ closed regions because the curves $\sigma^{\mathrm{u,s}}_{+,-}(p, \mu)$, $p = p^{+,-}(\theta_0)$, $\theta_0 = 0, \pi$, could have more intersections than the ones shown in the simplest case of Fig. 3.8a and 3.8b. However, the number of different possible qualitative behaviours is always given by this minimum number of regions.

Finally, we remark that the number of regions of the partition of S, and hence the number of qualitatively different orbits of (2.4), tends to infinity with k, that is, when $\alpha \downarrow 2$.

TABLE 3.1

Regions in S and the Corresponding Types of Trajectories. Case $F3(1)$

$R1$	E(+, u), 8, C(+, l)	$R14$	E(±, u), 2, C(±, l)
$R2$	E(+, l), 8, C(+, u)	$R15$	E(−, *u*), 3, C(−, u)
$R3$	E(+, u), 7, C(+, l)	$R16$	E(−, u), 3, C(−, l)
$R4$	E(+, l), 7, C(+, u)	$R17$	E(−, l), 4, C(−, u)
$R5$	E(+, u), 6, C(+, l)	$R18$	E(−, u), 4, C(−, l)
$R6$	E(+, l), 6, C(+, u)	$R19$	E(−, l), 5, C(−, u)
$R7$	E(+, u), 5, C(+, l)	$R20$	E(−, u), 5, C(−, l)
$R8$	E(+, l), 5, C(+, u)	$R21$	E(−, l), 6, C(−, u)
$R9$	E(+, u), 4, C(+, l)	$R22$	E(−, u), 6, C(−, l)
$R10$	E(+, l), 4, C(+, u)	$R23$	E(−, l), 7, C(−, u)
$R11$	E(+, l), 3, C(+, u)	$R24$	E(−, u), 7, C(−, l)
$R12$	E(+, u), 3, C(+, l)	$R25$	E(−, l), 8, C(−, u)
$R13$	E(±, l), 2, C(±, u)	$R26$	E(−, u), 8, C(−, l)

As an example, we describe in Table 3.1 the qualitative behaviour of the orbits for a flow of type $F3(1)$, see Fig. 3.8b. We have considered only the trajectories corresponding to points in the interior of regions R_j. For points on the boundary of R_j, we have already seen that the trajectories are as the ones of Fig. 3.9; that is, the assymptotic behaviour may be as in Fig. 3.6. Using the results obtained so far, it is easy to complete Table 3.1. For instance, given a point p in the boundary of regions R_1 and R_4, the corresponding trajectory will be of type $[\mathrm{E}(+,1), 7, \mathrm{c}(+,1)]$.

4. Generalizations and Final Remarks

The results of the previous sections may be applied directly when the function $f(r)$ is of the form $\sum_{x \in D} f^x(r)$, where $f^x(r)$ is homogeneous of degree $-x$ and D is a finite set contained in $(2, +\infty)$. In that case, the hypothesis that the energy parameter h is small enough may be weakened, and the results hold for every $h<0$. This family includes a large class of physical potentials.

In general case, we may substitute the hypothesis that h is small enough by supposing that $|xf(r)+rf'(r)| < -xh/|V(\theta)|$ in every point of I_h.

We have taken the perturbing function $V(\theta)$ as the natural generalization of the perturbation in the anisotropic Kepler problem. However, the results derived here apply directly when $V(\theta)$ is any C^2-function from S^1 to $\Re^-$ with four non-degenerate critical points. Similar results hold when $V: S^1 \to \Re^-$ is any function of class C^2 with a finite number of non-degenerate critical points.

We would like to stress that, when $f(r)$ is homogeneous, then system (2.1) is non-integrable for every $\mu > 1$ (this may be easily proved using a non-integrability criterium based on Ziglin's theorem, see [Z, Y]). Hence, Theorem 3.11 gives a complete qualitative classification of all the orbits of a large family of non-integrable Hamiltonians.

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